Selected Topics in Modern Many-Body Theory

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Notation and General Remarks

1. Throughout this note, we use the units of $\hbar \equiv 1$, unless restored explicitly when necessary.
2. Most presentations are derived without giving historic references. It does not mean we claim originality here, but some derivations and results are.
3. This is a course note meant to save students' time in taking notes from blackboard. It will be complemented by blackboard derivations and illustrations.
4. The note is far from being carefully checked yet. Appreciate anyone reporting typos, errors in derivations, etc. by email (w.vincent.liugmail.com) or directly in classroom.

1 Topological states in one dimension

1.1 Example: Kitaev quantum wire model

We shall introduce Kitaev's quantum wire model and use it an example to study topological numbers and one of the powerful mapping methods in 1D, Jordan-Wigner transformation.

Kitaev introduced an interesting 1D lattice quantum wire model of fermions (say, electrons with all spins polarized in one direction) that demonstrates the existence of topologically protected Majorana zero edge modes. The Hamiltonian reads

$$H = \sum_j \left[-w(c_j^\dagger c_{j+1} + h.c.) - \mu(c_j^\dagger c_j - \frac{1}{2}) + (\Delta c_j c_{j+1} + h.c.) \right] \tag{1}$$

where $j = -(L/2a) + 1, \ldots, (L/2a)$ with L the length of the wire and a the lattice constant, and hence L/a the total number of sites of the lattice. The U(1) symmetry is explicitly broken by a finite Δ, which in general is complex. The superconducting gap parameter Δ is thought to be arising from proximity effect, i.e., it is assumed that this 1D quantum wire is put next to a 3D superconductor with mutual electron tunneling. Without loosing generality, let us assume Δ real.

By Fourier transforming the fermion operators,

$$c_j = \frac{1}{\sqrt{L}} \sum_k e^{ikja} c_k \tag{2}$$

$$c_k = \frac{1}{\sqrt{L}} \sum_j e^{-ikja} c_j \tag{3}$$

the above Hamiltonian is diagonal in momentum space,

$$H = \sum_k \left[-(2w\cos(ka) + \mu)\, c_k^\dagger c_k + (i\Delta \sin(ka) c_{-k} c_k + h.c.) \right] + Const. \tag{4}$$

The Hamitonian is quadratic in the fermion operators. Using Nambu spinor

$$\psi_k \equiv \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}, \tag{5}$$

re-write the Hamiltonian

$$H = \frac{1}{2} \sum_k \psi_k^\dagger \mathscr{K}(k) \psi_k \tag{6}$$

where the $\mathscr{K}$ is a 2×2 matrix function of momentum k,

$$\mathscr{K}(k) = \begin{pmatrix} -(2w\cos(ka) + \mu) & -2i\Delta^* \sin(ka) \\ 2i\Delta \sin(ka) & (2w\cos(ka) + \mu)\,. \end{pmatrix} \tag{7}$$

In the basis of Pauli matricies, this Hamitonian $\mathscr{K}$ matrix may be represented by a vector $\mathbf{h}(k)$, defined by the matrix equation

$$\mathscr{K}(k) = \mathbf{h}(k) \cdot \boldsymbol{\sigma}\,, \quad \mathbf{h}(k) = (0, 2\Delta \sin(ka)\,, -2w\cos(ka) - \mu)\,. \tag{8}$$

Here, without loss of generality, we had chosen the phase of the gap parameter to be 0, so that the "Hamiltonian" $\mathbf{h}(k)$ vector is 2-component.

Here is an important note: for non-trivial topology (to be elaborated below), it is crucial that the two "spin" components of $\mathbf{h}(k)$ depend on the orthogonal basis functions, sin and cos, separately and both are periodic in k. For a general complex Δ^*, $\mathbf{h}$ is a 3-component vector. Nevertheless, one can verify that the presumed h^x component will denpend on k through $\sin(ka)$, exactly in the same manner as h^y. Thus, $\mathbf{h}$ can always be transformed into the above planar form (i.e., have one component eliminated), by a unitary "rotation" of the Pauli matrices, which is equivalent to a "Nambu-spin"-dependent rotation in the Nambu spinor space. Therefore, as far as k-functional dependence is concerned for topology purpose, it is perfectly fine to "rotate" the phase of the gap parameter Δ to 0.

The energy spectrum can be quickly obtained by finding the eigenvalues of the $\mathscr{K}$ matrix,

$$E_k = \pm\sqrt{(2w\cos(ka) + \mu)^2 + 4|\Delta|^2 \sin^2(ka)} \tag{9}$$

Half filling $\mu = 0$. To make a concrete discussion, let us consider half filling case, corresponding to the chemical potential $\mu = 0$. Verify that the lower and upper branches of the spectrum are well seperated by a gap of $4|\Delta|$. The fermi level lies in between (i.e., the line of zero energy, $\mu = 0$, in the k-space). The spectrum is fully gapped.

1.2 Topoogical winding number.

Again, for concreteness, study the case of $\mu = 0$. Verify that the $\mathbf{h}(k)$ vector has a non-vanishing magnitude in the whole mometum space, i.e., 1st Brillouin zone (1BZ), $k \in [-\pi/a, +\pi/a]$. Then, define a unit vector

$$\hat{\mathbf{h}}(k) \equiv \frac{\mathbf{h}(k)}{|\mathbf{h}(k)|}\,, \qquad (|\hat{\mathbf{h}}| = 1)\,. \tag{10}$$

A topological invariant (winding number in this case) for $\mathbf{h}$ is then expressed by

$$W = \oint_{1BZ} \frac{dk}{4\pi} \epsilon_{\alpha\beta} \hat{h}_\alpha^{-1} \frac{\partial \hat{h}_\beta}{\partial k}\,, \quad \text{indices } \alpha, \ldots = y, z \text{ two components} \tag{11}$$

and $\epsilon_{\alpha\beta}$ is a standard anti-symmetric tensor. The close contour integral means integration over a complete Brillouin zone in the momentum space.

The expression (11) is quite general for 1D quantum systems. W is always an interger. I will prove this in class on the blackboard when time permits. One can easily generalize such an expression for topological invariant to higher dimensions.

1.3 Domail wall (soliton) bound states. Fermion zero mode

Again consider the case of half filling $\mu = 0$. Recall the Hamiltonian $\mathscr{K}$ matrix

$$\mathscr{K}(k) = \begin{pmatrix} \epsilon(k) & -2i\Delta(x)\sin(ka) \\ 2i\Delta(x)\sin(ka) & \epsilon(k) \end{pmatrix} \tag{12}$$

where $\epsilon(k) = -2w\cos(ka)$.

We want to study the engenvalue and states of the Hamiltonian $\mathscr{K}$ matrix when the gap parameter is not a constant, but slowly varying in space:

$$\Delta \to \Delta(x) = \Delta_0 \tanh(x/\xi) \tag{13}$$

where Δ_0 is constant, and ξ is a large length scale characterizing the slowly varying gap function ($\xi \gg a$). Here "slow" means the gap function varies on the scale of several lattice constant or greater.

Here we demonstrate a "semiclassical" analytical approach to find the solution. The fermi level is set by $\epsilon(k) = \mu = 0$. This gives the two fermi points:

$$k_F^{L,R} = \mp k_F\,, \qquad k_F = \frac{\pi}{2a} \tag{14}$$

Let us restrict ourselves to the vicinity of the "Right" fermi point $k = +\pi/(2a)$. From the gap function, ξ^{-1} sets the low momentum scale (long wavelength limit), and we are studying physics of $|k - k_F| \ll \pi/\xi$. For this physical regime, expand the single particle spectrum around the Fermi point,

$$\begin{aligned} \epsilon(k) &\approx \epsilon(k_F) + 2w\sin(k_F a)\cdot ap\,, \qquad p \equiv k - k_F & (15) \\ &\approx (2wa)p & (16) \\ \Delta\sin(ka) &\approx \Delta\sin(k_F a) = \Delta \times 1 & (17) \end{aligned}$$

In the long wavelength limit, Fourier transform the algebraic Hamiltonian SK matrix back to the real coordinate space. Equivalently, we could quickly obtain the real space Hamiltonian matrix by

$$p \rightarrow -i\partial_x \tag{18}$$

$$\epsilon(k) \rightarrow (2wa)(-i\partial_x)\,, \qquad \text{(use units of } \hbar \equiv 1) \tag{19}$$

Then, and we aim to find the engenvalue and states when the gap parameter is not a constant, but slowly varying in space:

$$\mathscr{K} \rightarrow \hat{\mathscr{H}} = \begin{pmatrix} 2wa(-i\partial_x) & -2i\Delta(x) \\ 2i\Delta(x) & -2wa(-i\partial_x) \end{pmatrix} \tag{20}$$

$$= \sigma_z \cdot (-2iwa)\frac{\partial}{\partial_x} + \sigma_y \cdot 2\Delta(x) \tag{21}$$

The problem of eigenvalue and eigenfunctions is

$$\hat{\mathscr{H}} \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} = E_n \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} \tag{22}$$

Specifically, we are interested in finding out whether there is a physical zero mode solution (i.e., $E_{n=0} = 0$ solution):

$$\hat{\mathscr{H}} \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} = 0\,. \tag{23}$$

For the zero mode, there is a symmetry in exchanging $u \leftrightarrow v$. Note

$$\sigma_x \hat{\mathscr{H}} \sigma_x = -\hat{\mathscr{H}} \tag{24}$$

If $\binom{u(x)}{v(x)}$ is a solution of $E = 0$, then so is $\sigma_x \binom{u(x)}{v(x)}$:

$$\hat{\mathscr{H}} \sigma_x \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} = -\sigma_x \hat{\mathscr{H}} \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} = 0\,. \tag{25}$$

The two math vectors must correspond to the same physical state. That means they can only differ by a complex phase (after normalization),

$$\begin{pmatrix} v(x) \\ u(x) \end{pmatrix} = e^{i\theta} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}, \tag{26}$$

Solving this matrix equation gives

$$e^{i2\theta} = 1 \Rightarrow \theta = 0, \text{or } \pi. \tag{27}$$

There are two possible solutions:

Case of $u = v$.

$$\left[-i2wa\frac{\partial}{\partial x} - i2\Delta(x)\right] u(x) = 0 \tag{28}$$

$$\frac{1}{u}\frac{\partial u}{\partial x} = -\frac{\Delta_0}{wa}\tanh\frac{x}{\xi} \tag{29}$$

Solution is found

$$u(x) = u(0)\left[\cosh\frac{x}{\xi}\right]^{-\frac{\Delta_0\xi}{wa}} \tag{30}$$

This is a zero energy fermion state localized around $x = 0$ (the location of the domail wall), whose wavefunction decays exponetially

$$e^{-|x|/\lambda}, \quad \lambda = \xi \cdot \frac{wa}{\Delta_0\xi} = \frac{w}{\Delta_0}a \tag{31}$$

as $x \to \pm\infty$.

Case of $u = -v$. For this case, the solution can be found in the similar manner (by $\Delta_0 \to -\Delta_0$ from symmetry). We get

$$u(x) = u(0)\left[\cosh\frac{x}{\xi}\right]^{+\frac{\Delta_0\xi}{wa}} \tag{32}$$

This "mathematical" solution to the above differential equation diverges at large distance ($x \to \pm\infty$), noticing the exponent is positive. This wavefunction is *not normalizable, hence non-physical.*

In conclusion, we find *only one physical fermion zero mode.*

Exercise 1: Expand the Hamiltonian matrix $\mathscr{K}$ around the 'left' Fermi point. Find the 'left-moving' mode of the zero-energy bound state. Prove that the eigenwavefunctions take the exactly same expression as $(u(x), v(x))$ for the 'Right' moving mode.

1.4 Majorana fermion.

From lattice to continuum limit, the lattice fermion annihilation operator may be expressed by the 'left' (L) and 'right' (R) moving modes, with respect to the left and right Fermi surfaces (points in 1D),

$$\frac{c_j}{\sqrt{a}} = \hat{\psi}_R(x_j)e^{ik_Fx_j} + \hat{\psi}_L(x_j)e^{-ik_Fx_j}\,. \tag{33}$$

[D. Senechal, "Introduction to Bosonization", cond-mat/9908262, Sec. 3.1, Eq. (14)]. Each of the left and right moving fermion fields $\hat{\psi}$ captures the dynamics of the *slowly varying* modes in the space, with the factors $e^{\pm ik_F x}$ extracted for the *fast* oscillating background.

In terms of original "right" moving fermion operators, the zero mode obtained above is expressed as follows

$$\begin{aligned} \gamma_R(x) &= u(x)\hat{\psi}_R(x)e^{ik_F x} + v(x)\hat{\psi}_R^\dagger(x)e^{-ik_F x} \\ &= u(x)[\hat{\psi}_R(x)e^{ik_F x} + \hat{\psi}_R^\dagger(x)e^{-ik_F x}]\,, \quad (\text{by } u = v) \end{aligned} \tag{34}$$

In a similar spirit, the left-moving zero mode of the quasiparticle is expressed by the operator

$$\gamma_L(x) = u(x)[\hat{\psi}_L(x)e^{-ik_F x} + \hat{\psi}_R^\dagger(x)e^{+ik_F x}]\,, \tag{35}$$

(by $u_L = u_R$, $u = v$ for both left and right movers). The full fermionic quasi-particle of zero energy combines the left and right moving modes,

$$\gamma(x) = \gamma_R + \gamma_L = \frac{u(x)}{\sqrt{a}}(c_x + c_x^\dagger) \tag{36}$$

where the last equation was obtained by using (33). One can immediately verify

$$\gamma(x_j)^\dagger = \gamma(x_j)\,, \quad x_j = ja, \quad (j \text{ labels lattice sites}) \tag{37}$$

In summary, the operator $\gamma(x)$ is Majorana; it decribes a fermionic zero-energy state bound to the domail wall (soliton) at $x = 0$, with a wavepacket size characterized by λ in (31).

2 Effective field theory approach to superconductivity

2.1 Renormalization group analysis of Fermi surface instability: Polchinski approach

[Ref: J. Polchinski, hep-th/9210046]

Consider a quantum gas of spin-1/2 fermions of mass m whose dispersion is assumed to take the simple free-particle form: $\varepsilon(\mathbf{p}) = \mathbf{p}^2/2m$, for both spin $\sigma =\uparrow, \downarrow$. The field theory is described by the action

$$S_0 = \int dt \int d^3\mathbf{p} \sum_\sigma \psi^\dagger_\sigma(\mathbf{p}, t)[i\partial_t - \varepsilon(\mathbf{p}) + \mu]\psi_\sigma(\mathbf{p}, t)\,, \tag{38}$$

The chemical potential specifies the location of Fermi surfaces in momentum space, with Fermi momentum

$$k_F = \sqrt{2m\mu}\,, \quad \text{by} \quad \frac{k_F^2}{2m} = \mu\,. \tag{39}$$

Note that an important energy quantity, Fermi energy, may be defined as the chemical potential at zero temperature for a free fermi gas: $E_F = \mu(T = 0)$. (E_F, k_F) are the charactristic energy and momentun scales of this system.

In the spirit of low energy effective field theory, such that all physical processes that we are interested in are much below E_F, i.e.,

$$E \ll E_F, \quad |\mathbf{p} - \mathbf{k}_F| \ll k_F$$

where $|\mathbf{k}_F| = k_F$. Then, we can expand the energy spectrum along the Fermi surface, i.e., at $|\mathbf{p}| = k_F$:

$$\mathbf{p} = k_F\mathbf{\Omega} + \mathbf{l}\,, \qquad \mathbf{l} \perp \text{Fermi surface} \tag{40}$$

where $\mathbf{\Omega}$ is a unit vector (solid angle) that parameterizes the Fermi surface of the up-fermions. For a spherical Fermi surface, $\mathbf{\Omega}$ is parallel to $\mathbf{p}$, but it is not necessarily true for general case.

The free part of the effective theory becomes

$$\int dt \int d^2\Omega \int_{-\Lambda}^{\Lambda} dl \sum_{\sigma=\uparrow,\downarrow} \psi^\dagger_\sigma[i\partial_t - v_F l]\psi_\sigma \tag{41}$$

where

$$v_F = |\nabla_{\mathbf{p}}\varepsilon_(\mathbf{p})|_{|\mathbf{p}|=k_F}$$

and $\Lambda \ll k_F$ is a *high-energy cutoff* (Students may consider this equivilent to the *ultra-violet cutoff* in particle physics).

Sliding momentum scale - spirit of RG. Now let us slide down the momentum scale of physical processes:

$$\kappa \to s\kappa\,, \quad (s \to 0 \quad \text{means low energy, long wavelength limit}) \tag{42}$$

where κ is our characteristic energy scale and s the rescaling ratio. Long wavelength limit (low energy limit) corresponds to $s \to 0$.

Scaling transformation. The action is invariant under the following scaling transformation:

$$dt \to s^{-1}dt, \quad d\Omega \to s^0\, d\Omega, \quad dl \to sdl, \quad \partial_t \to s\partial_t, \quad l \to sl, \quad \psi_\sigma \to s^{-\frac{1}{2}}\psi_\sigma\,. \tag{43}$$

Note that the linear spectrum is very important for both the inverse of time and the longitudinal momentum $\mathbf{l}$ to share the same scaling dimension 1. This case corresponds to the so-called *dynamical exponent* $z = 1$.

Turn on a generic two-body scattering between up- and down-fermions

$$\begin{aligned} S_{int} &= \int dt \prod_{i=1}^{4} \left[\int d^2\Omega_i \int_{-\Lambda}^{\Lambda} dl_i\right] V_{\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4} \\ &\quad \times\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)\psi^\dagger_\uparrow(\mathbf{p}_1)\psi^\dagger_\downarrow(\mathbf{p}_2)\psi_\downarrow(\mathbf{p}_3)\psi_\uparrow(\mathbf{p}_4) \end{aligned} \tag{44}$$

with $\mathbf{p}_i \equiv k_F\mathbf{\Omega}_i + \mathbf{l}_i$. Now let us analyze how the interaction V should scale when applying the scaling transformation (43). Some basic power counting algebra gives

$$V_{\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4} \to V_{\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4}\, s^{-1+4\times2\times0+4\times1+\Delta_\delta+4\times(-1/2)} \sim V_{\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4}\, s^{1+\Delta_\delta} \tag{45}$$

where Δ_δ is the scaling dimension of the δ-function, to be examined in detail next.

For a scattering process with general incoming and outgoing momenta, the sum of

$$\mathbf{\Omega}_1 + \mathbf{\Omega}_2 - \mathbf{\Omega}_3 - \mathbf{\Omega}_4\,, \tag{46}$$

is an arbitary 3-dimension vector in the momentum space and the perpendicular components $\mathbf{l}_i's$ are expected all samll in magnitude compared with k_F. Therefore, the delta function reduces to

$$\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \to \delta^3(\mathbf{\Omega}_1 + \mathbf{\Omega}_2 - \mathbf{\Omega}_3 - \mathbf{\Omega}_4) \tag{47}$$

for all $\mathbf{l}_i/k_F \ll 1$ and so $\mathbf{l}$'s in the delta function are negligible. An important consequence is that the δ-function in the integral does not restrict the integration of $\mathbf{l}$'s, so not affecting the scaling. That is, for this generic scattering process,

$$\Delta_\delta = 0\,. \tag{48}$$

Therefore, the interaction (44) for such a general case scales

$$V_{\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4} \to s^1 V_{\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4} \tag{49}$$

when $s \to 0$ approaching long wavelength limit. Such an interaction is called *irrelevant* in RG analysis. It is easy to check that field operators of high powers of ψ's (e.g., three-body interaction with 6 field operators) are even more irrelevant.

BCS scattering channel and instability However, a special situation arises when the sum vector (46) has a dimension lower than 3, hence the δ-function may restrict the integral of $\mathbf{l}$'s. For example, consider the process satisfying with the following condition:

$$\boldsymbol{\Omega}_1 \parallel -\boldsymbol{\Omega}_2\,, \quad \boldsymbol{\Omega}_3 \parallel -\boldsymbol{\Omega}_4\,,$$

(BCS channel, momenta anti-parallel). One immediately sees that the sum vector (46) is now actually co-planar (i.e., all 4 Ω's lie in a common plane), and has only two independent components in a 3D vectorial space. Therefor the sum of the 4 vectors is a 2-dimensional vector in nature. Then, the small momenta, $\mathbf{l}$'s, in $\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$ cannot be neglected. The 3-dimensional δ-function restricts the integration over $\boldsymbol{\Omega}$'s in the 2-dim co-plan and restricts by *one dimension* the integration over dl's, so that its scaling dimension is

$$\Delta_\delta = -1 \tag{50}$$

Inserting the scaling dimension into (45) gives

$$V_{\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4} \to s^{1+(-1)} V_{\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4} = s^0 V_{\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4}\,. \tag{51}$$

In other words, the interaction has scaling dimension 0 and is called *marginal* in RG. The above argument may be formally summarized by

$$\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \to \delta^2(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_3 - \boldsymbol{\Omega}_4) \times \delta(\mathbf{l}_1 + \mathbf{l}_2 - \mathbf{l}_3 - \mathbf{l}_4)\,.$$

It is marginal in the tree level so far, as we have not included the correction to the interaction yet, due to quantum scattering.

One-loop renormalization of interaction in BCS channel. When the interaction is classically (tree-level) marginal, it is important to look at quantum corrections. Assume a constant $V_{\mathbf{p},-\mathbf{p};-\mathbf{p}',\mathbf{p}'} = V$. To the one-loop order, the scattering process involving a pair of up and down fermions with $(\mathbf{p}_1, E)$ and $(-\mathbf{p}_1, E)$ gives a correction to the coupling constant,

$$\begin{aligned} V' &= V - (\mathrm{vol})V^2 \int \frac{d\omega\, k_F^2 d^2\Omega dl}{i(2\pi)^4} \\ &\quad \times \left(\frac{1}{\omega - v_F l + i\omega\eta} \right) \left(\frac{1}{2E - \omega - v_F l + i(2E-\omega)\eta} \right) \end{aligned} \tag{52}$$

where $\eta = 0^+$ is infinitesimal and positive definite.

As far as low energy process is concerned, we consider all external momenta $\sim \kappa$ and have external energies $E \sim v_F \kappa$. Define our renormalized interaction at this scale, V'. After performing the integration over ω,

$$\begin{aligned} V' &= V - V^2 \frac{(\mathrm{vol})}{(2\pi)^3} k_F^2 \int d^2\Omega \frac{1}{2v_F} 2 \int_0^\Lambda dl \frac{1}{l-\kappa} \\ &= V + V^2 N(k_F) \ln\left(\frac{\kappa}{\Lambda}\right) + O(\kappa/\Lambda) \end{aligned} \tag{53}$$

where

$$N(k) \equiv \frac{(\mathrm{vol})}{(2\pi)^3} \int d^2\Omega \frac{k^2}{v_F} \tag{54}$$

is the density of states.

RG flow equation can be obtained by differentiating (53) with respect to κ,

$$\kappa \frac{dV}{d\kappa} = V^2 N(k_F) \tag{55}$$

which gives

$$V_\kappa = \frac{V_0}{1 - N(k_F) V_0 \ln(\kappa/\kappa_0)}\,, \qquad \kappa_0 \sim \Lambda\,. \tag{56}$$

So an attractive interaction, $V < 0$, grows stronger at low energy. We conclude that the RG flow is rather conventional for interacting fermions having a Fermi surface.

2.2 BCS superconductivity: fermionic excitations

In this section, we shall work in finite temperature (Euclidean) path-integral framework. In order to keel notation compact (to be seen below), we will put the system in a finite box of liniear dimension L and take the thermodynamic limit $L^3 \to 0$ after performing all derivations at each calculation.

Let us start with the many-body theory with the free and interacting action described in Eqs. (38) and (44). From real to imaginary time formalism, make the relation:

$$it \to \tau, \qquad i\partial_t \to -\partial_\tau, \qquad S \to -S_E \tag{57}$$

We further focus on the BCS channel:

$$\mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}\,, \qquad \mathbf{p}_3 = -\mathbf{p}_4 \equiv \mathbf{p}'\,,$$

and subsitute the interaction by the following form:

$$V_{\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4} = V(\mathbf{p}, \mathbf{p}') \tag{58}$$

The Euclidean (imaginary time) action reads

$$\begin{aligned} S &= \int d\tau \sum_{\mathbf{p},\sigma} \psi_\sigma^\dagger(\mathbf{p},\tau)[\partial_\tau + \varepsilon(\mathbf{p}) - \mu]\psi_\sigma(\mathbf{p},\tau) \\ &\quad + \int d\tau \sum_{\mathbf{p}\mathbf{p}'} V(\mathbf{p},\mathbf{p}')\psi_\uparrow^\dagger(\mathbf{p})\psi_\downarrow^\dagger(-\mathbf{p})\psi_\downarrow(-\mathbf{p}')\psi_\uparrow(\mathbf{p}') \end{aligned} \tag{59}$$

where the time variable is suppressed in the interaction part to keep notation simple. The interaction being Hermitian requires

$$V(\mathbf{p},\mathbf{p}')^* = V(\mathbf{p}',\mathbf{p}) \tag{60}$$

This theory is invariant under a global U(1) symmetry transformation,

$$\psi_\sigma \to e^{ie\theta}\psi_\sigma \tag{61}$$

Here we assume the fermion carry "electric" charge e. The attactive interaction V induces Cooper pairing between two fermions and when the pairs condense, superconductivity occurs.

Exercise 2: Check whether the theory defined by the above BCS action is spin SU(2) invariant.

Order parameter Let Φ denote the quantum average (ground-state expectation value) of the composite pair field operator,

$$\Phi^*(\mathbf{p}) = \frac{1}{2}\epsilon_{\sigma\sigma'}\langle\psi_\sigma^\dagger(\mathbf{p})\psi_{\sigma'}^\dagger(-\mathbf{p})\rangle = \langle\psi_\uparrow^\dagger(\mathbf{p})\psi_\downarrow^\dagger(-\mathbf{p})\rangle \tag{62}$$

with $\epsilon_{\sigma\sigma'}$ an antisymmetric tensor:

$$\epsilon_{\uparrow\downarrow} = -\epsilon_{\downarrow\uparrow} = 1\,, \qquad \epsilon_{\uparrow\uparrow} = \epsilon_{\downarrow\downarrow} = 0\,. \tag{63}$$

The $\langle \ldots \rangle$ here is over the many-body ground state. We shall not derive the equation that will govern the value of this order parameter, but instead we will assume the pair operator has acquired some finite mean field value. We take Φ to be finite as our starting point for the derivation below and would like to explore the properties due to the appearance of this order.

When the order parameter Φ aquires a non-zero value,

$$\Phi(\mathbf{p}) \neq 0\,,$$

it signals that the U(1) symmetry is spontaneously broken. Under the U(1) tranformation,

$$\Phi \to e^{i2e\theta}\Phi \tag{64}$$

so this quantity is not invariant. If the U(1) symmetry were not broken, it would have required this quantity to be zero. "Spontaneous" means the symmetry is broken by the ground state, in this case, it is referred to as BCS many-body wavefunction (not derived here). A very important point is needed: the U(1) symmetry is never broken in the Hamiltian or Lagrangian level.

Mean field theory Once the fermions pair and condense, the order parameter Φ aquires mean field value. The mean-field-theory action can then be obtained by decomposing the quartic interaction term into quandratic terms,

$$\begin{aligned} S \;=\; & \int d\tau \sum_{\mathbf{p},\sigma} \psi^\dagger_\sigma(\mathbf{p},\tau)[\partial_\tau + \varepsilon(\mathbf{p}) - \mu]\psi_\sigma(\mathbf{p},\tau) \\ & + \int d\tau \sum_{\mathbf{p}\mathbf{p}'} V(\mathbf{p},\mathbf{p}')\langle \psi^\dagger_\uparrow(\mathbf{p})\psi^\dagger_\downarrow(-\mathbf{p})\rangle \psi_\downarrow(-\mathbf{p}')\psi_\uparrow(\mathbf{p}') \\ & + \int d\tau \sum_{\mathbf{p}\mathbf{p}'} V(\mathbf{p},\mathbf{p}')\psi^\dagger_\uparrow(\mathbf{p})\psi^\dagger_\downarrow(-\mathbf{p})\langle \psi_\downarrow(-\mathbf{p}')\psi_\uparrow(\mathbf{p}')\rangle + \cdots \qquad (65) \\ =\; & \int d\tau \sum_{\mathbf{p},\sigma} \psi^\dagger_\sigma(\mathbf{p},\tau)[\partial_\tau + \varepsilon(\mathbf{p}) - \mu]\psi_\sigma(\mathbf{p},\tau) \\ & + \int d\tau \sum_{\mathbf{p}\mathbf{p}'} V(\mathbf{p},\mathbf{p}') \left[\Phi^*(\mathbf{p})\psi_\downarrow(-\mathbf{p}')\psi_\uparrow(\mathbf{p}') + \psi^\dagger_\uparrow(\mathbf{p})\psi^\dagger_\downarrow(-\mathbf{p})\Phi(\mathbf{p}')\right] + \cdots \qquad (66) \end{aligned}$$

where $\cdots$ stand for other terms not needed to study the fermion excitation at the mean field level.

Energy gap

$$\begin{aligned}\Delta(\mathbf{p}) &= \sum_{\mathbf{p}'} V(\mathbf{p},\mathbf{p}')\Phi(\mathbf{p}') \\ \Delta^*(\mathbf{p}') &= \sum_{\mathbf{p}} V^*(\mathbf{p}',\mathbf{p})\Phi^*(\mathbf{p}) = \sum_{\mathbf{p}} V(\mathbf{p},\mathbf{p}')\Phi^*(\mathbf{p}) \end{aligned} \tag{67}$$

Using the gap functions introduced above in the mean field action, we get

$$\begin{aligned} S_{MFT} &= \int d\tau \sum_{\mathbf{p},\sigma} \psi^\dagger_\sigma(\mathbf{p},\tau)[\partial_\tau + \varepsilon(\mathbf{p}) - \mu]\psi_\sigma(\mathbf{p},\tau) \\ &\quad + \int d\tau \sum_{\mathbf{p}} \left[\Delta^*(\mathbf{p})\psi_\downarrow(-\mathbf{p})\psi_\uparrow(\mathbf{p}) + \Delta(\mathbf{p})\psi^\dagger_\uparrow(\mathbf{p})\psi^\dagger_\downarrow(-\mathbf{p})\right] \end{aligned} \tag{68}$$

In the imaginary time formalism, the time space is $[0,\beta]$ where $\beta = 1/(k_B T)$ is the inverse temperature (set $k_B = 1$). The action can be rewritten in frequency space by performing the Fourier transformation on the fermion fields,

$$\psi_\sigma(\mathbf{p},\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} e^{-i\omega_n \tau}\psi_\sigma(\mathbf{p}, i\omega_n) \tag{69}$$

where ω_n is the Matsubara frequency,

$$\omega_n = \frac{\pi(2n+1)}{\beta}, \qquad \beta = \frac{1}{k_B T}, \qquad n = 0, \pm1, \pm2, \cdots \tag{70}$$

Further intruduce a Nambu spinor

$$\Psi(\mathbf{p}, i\omega) = \begin{pmatrix} \psi_\uparrow(\mathbf{p}, i\omega) \\ \psi^\dagger_\downarrow(-\mathbf{p}, -i\omega) \end{pmatrix} \tag{71}$$

The action can be written in a matrix form

$$S_{MFT} = \sum_{\mathbf{p},\omega_n} \Psi^\dagger(\mathbf{p}, i\omega_n)\mathscr{K}(\mathbf{p}, i\omega_n)\Psi(\mathbf{p}, i\omega_n)\,, \tag{72}$$

with

$$\mathscr{K}(\mathbf{p}, i\omega_n) = \begin{pmatrix} -i\omega_n + \varepsilon(\mathbf{p}) - \mu & \Delta(\mathbf{p}) \\ \Delta^*(\mathbf{p}) & -i\omega_n - \varepsilon(\mathbf{p}) + \mu \end{pmatrix} \tag{73}$$

Fermionic quasi-particle spectrum The quasiparticle energy spectrum can be obtained from the K matrix by the following procedure: 1) Wick rotating back to real frequency axis,

$$i\omega_n \to \omega + i\eta$$

2) Determine the zeros

$$\det \mathscr{K} = 0$$

and find the solution for ω. Then find the energy spectrum

$$E(\mathbf{p}) = \pm\sqrt{(\varepsilon(\mathbf{p}) - \mu)^2 + |\Delta(\mathbf{p})|^2} \tag{74}$$

2.3 Green's function, mometum distribution, etc.

In the imaginary time formalism, the Green's function (also frequently referred to as *propagator* in quantum field theory/particle physics) is defined as

$$G_{ab}(\mathbf{p}, \tau - \tau') = -\langle T\Psi_a(\mathbf{p}, \tau)\Psi_b^\dagger(\mathbf{p}, \tau')\rangle \tag{75}$$

where the indices $a, b = 1, 2$ span the 2-dimensional Nambu spinor space, and T denotes a time-ordered product. For example, $G_{11} = -\langle T\psi_\uparrow\psi_\downarrow^\dagger\rangle$. In path-integral formalism, the time-order is implicit, and one does not need to keep this time-order operator at all.

In momentum-frequency space, the Green's function matrix reads

$$G_{ab}(\mathbf{p}, i\omega) = -\langle\Psi_a(\mathbf{p}, i\omega)\Psi_b^\dagger(\mathbf{p}, i\omega)\rangle \tag{76}$$

(Here we work in the path-integral formalism and the fields are Grassman variables now.)

Computation of G The Green's function matrix is read off from the inverse of the $\mathscr{K}$ matrix [let $p \equiv (\mathbf{p}, i\omega_n)$],

$$\begin{aligned} G(p) = -\mathscr{K}^{-1}(p) &= -\frac{1}{(i\omega_n - E_\mathbf{p})(i\omega_n + E_\mathbf{p})}\begin{pmatrix} -i\omega_n - \varepsilon(\mathbf{p}) + \mu & -\Delta^*(\mathbf{p}) \\ -\Delta(\mathbf{p}) & -i\omega_n + \varepsilon(\mathbf{p}) - \mu \end{pmatrix} \\ &= \frac{1}{(i\omega_n - E_\mathbf{p})(i\omega_n + E_\mathbf{p})}\begin{pmatrix} i\omega_n + \varepsilon_(\mathbf{p}) - \mu & \Delta^*(\mathbf{p}) \\ \Delta(\mathbf{p}) & i\omega_n - \varepsilon(\mathbf{p}) + \mu \end{pmatrix} \end{aligned} \tag{77}$$

Use of G The Green's functions are powerful tools to obtain experimentally measurable physical quantities. Here we show how to obtain the particle momentum distribution (occupation number), $n_\sigma(\mathbf{p})$, for the orginal spin σ fermions (not the quasiparticles).

Take $n_\uparrow(\mathbf{p})$ as example.

$$
\begin{aligned}
n_\uparrow(\mathbf{p}) &= \langle \psi_\uparrow^\dagger(\mathbf{p},\tau)\psi_\uparrow(\mathbf{p},\tau)\rangle \\
&= -\frac{1}{\beta}\sum_{\omega_n}\langle \psi_\uparrow(\mathbf{p},i\omega)\psi_\uparrow^\dagger(\mathbf{p},i\omega)\rangle e^{i\omega\eta} \\
&= -\frac{1}{\beta}\sum_{\omega_n}(-)G_{11}e^{i\omega\eta} \\
&= \frac{1}{\beta}\sum_{\omega_n}\frac{i\omega_n+\varepsilon_{\mathbf{p}}-\mu}{(i\omega_n-E_{\mathbf{p}})(i\omega_n+E_P)}e^{i\omega_n\eta} \\
&= \frac{1}{2\pi i}\oint_C \frac{\omega+\varepsilon_{\mathbf{p}}-\mu}{(\omega-E_{\mathbf{p}})(\omega+E_{\mathbf{p}})}e^{\omega\eta}f(\omega)
\end{aligned}
\tag{78}
$$

where C is a contour in the complex ω-plane encircling all the poles of the integrand except the factor of $f(\omega)$ in a positive sense, and $f(\omega)=\frac{1}{e^{\beta\omega}+1}$ is the Fermi-Dirac distribution.

$$
\begin{aligned}
n_\uparrow(\mathbf{p}) &= \frac{1}{2}\left(1+\frac{\varepsilon_{\mathbf{p}}-\mu}{E_{\mathbf{p}}}\right)f(E_{\mathbf{p}})+\frac{1}{2}\left(1-\frac{\varepsilon_{\mathbf{p}}-\mu}{E_{\mathbf{p}}}\right)f(-E_{\mathbf{p}}) \\
&= u_{\mathbf{p}}^2 f(E_{\mathbf{p}})+v_{\mathbf{p}}^2 f(-E_{\mathbf{p}})
\end{aligned}
\tag{79}
$$

where

$$
u_{\mathbf{p}}^2=\frac{1}{2}\left(1+\frac{\varepsilon_{\mathbf{p}}-\mu}{E(\mathbf{p})}\right), \qquad v_{\mathbf{p}}^2=1-u_{\mathbf{p}}^2 \tag{80}
$$

Exercise 3: Calculate $n_\downarrow(\mathbf{p})$.

2.4 Weinberg Effective Field Theory approach to broken U(1) symmetry and superconductivity

[Refs.: S. Weinberg, Theory of Quantum Fields, Vol 2, 1996, §21.6; more detailes in his earlier article, Prog. Theor. Phys. Suppl. No. 86, 43 (1986).]

Consider a microscopic model of spin-1/2 fermions, $\psi_\sigma(\mathbf{x},t)$, of mass m interacting with each other by a short-range (δ-like) potential. The Lagrangian is

$$L = \int d^3x \psi_\sigma^\dagger i\partial_t \psi_\sigma - H \tag{81}$$

$$= \int d^3x \psi_\sigma^\dagger \left[i\partial_t + \frac{\nabla^2}{2m} + \mu \right] \psi_\sigma - g \int d^3x \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \tag{82}$$

where the coupling constant is assumed attractive, $g < 0$.

Now further assume the fermions carry charge e and the electric and magnetic gauge field $A^\mu = (\phi, \mathbf{A})$ are present, where ϕ and $\mathbf{A}$ are the scalar and vector potential, in a standard convention. (In our convention, electrons have charge $e = -|e| < 0$.) The Lagrangian now becomes

$$L[\psi, A] = \int d^3x \psi_\sigma^\dagger \left[(i\partial_t + e\phi) + \frac{(\nabla - ie\mathbf{A})^2}{2m} + \mu \right] \psi_\sigma - g \int d^3x \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \tag{83}$$

This theory has the charge U(1) and spin SU(2) symmetry. For example, one can verify that the Lagrangian is invariant under the local U(1) gauge transformation defined as

$$\psi_\sigma(x) \rightarrow \psi'_\sigma(x) = \psi_\sigma(x) e^{ie\lambda(x)} \tag{84}$$

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\lambda \tag{85}$$

$$\phi \rightarrow \phi' = \phi + \partial_t \lambda \tag{86}$$

where $x \equiv (t, \mathbf{x})$ is a 4-component vector.

Here is an important starting point in our approach. We shall take a superconducting state as our starting point. The physical reason is of course that the RG analysis in the preceding section showed that any attractive interaction, no matter how weak at initial condition, shall drive the system to a BCS paired state. Take the superconducting order parameter field

$$\Delta(x) = -g\langle \psi_\uparrow^\dagger(x) \psi_\downarrow^\dagger(x) \rangle \tag{87}$$

and study the case when it has a non-vanishing amplitude, Δ_0, to be assume constant below. In general, the order parameter field can be written as

$$\Delta(x) = \Delta_0 e^{i2e\theta(x)} \tag{88}$$

where $\theta(x)$ is the phase field and $2e$ is because of the fermion pair for Δ.

With the superconducting order takes place, we can replace the fermion pair field product in the Lagrangian by its mean field value, plus quantum fluctuations of the order parameter field. Keeping terms coupled to fermions, we find the eff Then,

$$\begin{aligned} L[\psi, A, \theta] &= \int d^3x \psi^\dagger_\sigma \left[(i\partial_t + e\phi) + \frac{(\nabla - ie\mathbf{A})^2}{2m} + \mu \right] \psi_\sigma \\ &\quad + \int d^3x \left[\Delta_0 e^{-i2e\theta(x)} \psi_\downarrow \psi_\uparrow + h.c. \right] \end{aligned} \tag{89}$$

The fermion fields see the phase field through an off-diagonal potential, which is non-uniform for a generic phase variable in space and time. We know the θ field must disappear in the effective theory if it becomes space-time independent, due to the U(1) symmetry. So far, this is not obvious from the Lagrangian above.

'Rotate' the fermion field at each space and time point as follows, so to define a new fermion field,

$$\psi_\sigma(x) = \tilde{\psi}_\sigma e^{ie\theta(x)} . \tag{90}$$

This 'rotation' is designed to eliminate the phase θ depedence in the off-diagnal potential term. The phase field re-appear in the transformed theory through terms of space-time derivatives. Therefore, we obtain

$$\begin{aligned} L[\tilde{\psi}, A, \theta] &= \int d^3x \tilde{\psi}^\dagger_\sigma \left[i\partial_t + e(\phi - \partial_t\theta) + \frac{[\nabla - ie(\mathbf{A} - \nabla\theta)]^2}{2m} + \mu \right] \tilde{\psi}_\sigma \\ &\quad + \int d^3x \left[\Delta_0 \tilde{\psi}_\downarrow \tilde{\psi}_\uparrow + h.c. \right] . \end{aligned} \tag{91}$$

The U(1) symmetry transformation works on the new fields as follows,

$$\tilde{\psi}_\sigma(x) \rightarrow \tilde{\psi}'_\sigma(x) = \tilde{\psi}_\sigma(x) \tag{92}$$
$$\Delta_0 \rightarrow \Delta'_0 = \Delta_0 \tag{93}$$
$$\theta \rightarrow \theta' = \theta + \lambda \tag{94}$$
$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\lambda \tag{95}$$
$$\phi \rightarrow \phi' = \phi + \partial_t\lambda \tag{96}$$

From what we have learned in the last section, a finite superconducting order parameter Δ_0 (the gap parameter in the above) opens a gap in the fermionic excitation. One then can integrate out all gapped fermionic degrees of freedom in the path-integral formalism, that is, to find a low energy effective field theory for the supercinductor, valid for the enegy scale below

Δ_0. Then, the effective Lagrangian for the gauge and phase fields is found to take the following general form

$$\begin{aligned} L_{eff}[A,\theta] &= \int d^3x \Big\{ en_f(\phi - \partial_t\theta) + e\langle \mathbf{J}\rangle \cdot (\mathbf{A} - \nabla\theta) - \frac{n_f e^2}{2m}(\mathbf{A} - \nabla\theta)^2 \\ &\qquad -C_2^t(\phi - \partial_t\theta)^2 + \ldots \Big\} \end{aligned} \quad (97)$$

where $n_f = \langle \tilde{\psi}_\sigma^\dagger \tilde{\psi}_\sigma \rangle = \langle \psi_\sigma^\dagger \psi_\sigma \rangle$ is the fermion number density, $\mathbf{J}$ is the fermion number current, and ... standard for higher powers of derivatives. A few remarks are in order. First, in our approach (i.e., rotate fermion fields), the first 3 terms can be obtained without the need of real, hard calculation (not to involve calculation of integrals but basic algebra). They appear in Hartree-Fock level. Second, the current average is taken over with respect to the fermion action, and because all fermion modes are gapped, there is absolutely no current flow in the ground state (T=0). At finite temperature, it can be a small amount. Third, the 4th term comes from "vacuum polarization" of a pair of particle and hole, and it is equivalent to a one-loop Feynman diagram. I will leave out the detail, but only state that the coefficient C_2 is positive. Fourth, the phase field can only enter the effective theory through the ingredients $(\phi - \partial_t\theta), (\mathbf{A} - \nabla\theta)$ and their powers. This is required by the U(1) symmetry, because only such combinations are invariant under the U(1) transformation of the new fields.

Exercise 4: The fermion current is $\mathbf{J} = -\frac{i}{m}[\tilde{\psi}_\sigma^\dagger \nabla \tilde{\psi}_\sigma - (\nabla \tilde{\psi}_\sigma^\dagger)\tilde{\psi}_\sigma]$ Show that $\langle \mathbf{J} \rangle_\psi = 0$ as $\tilde{\psi}$ is gapped. This is sometime called *paramagnetic current.*

A number of important properties follow from the effective theory (97).

Flux quantization Deep inside the superconductor, expect no boundary effect, the system is at energy minimum. Note the energy is the minus of the Lagrangian. Minimizing the quadratic term $(\mathbf{A} - \nabla\theta)^2$ gives

$$\mathbf{A} = \nabla\theta \qquad \text{(known as } \textit{pure gauge}\text{)} \quad (98)$$

Consider a closed path C enbedded deep inside the superconductor. The magnetic flux through it is

$$\iint_S \mathbf{B} \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l} = \oint_C \nabla\theta \cdot d\mathbf{l} = \Delta\theta \quad (99)$$

From the definition of θ in the order parameter (88), we must have $(2e/\hbar)\Delta\theta = 2\pi n$ with $n \in$ integer. Therefore,

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \frac{n\pi\hbar}{e}, \qquad n = 0, \pm1, \pm2, ... \quad (100)$$

Here I have restored $\hbar$.

Meissner effect. London peneration depth. Now consider somewhere near the surface of a superconductor, and in this case the magnetic field is expected not vanishing to a certain depth, known as London penetration delpth. Here we show how to quickly derive it in our approach.

Redefine the gauge field by a U(1) gauge transformation, so to absorb the phase dependence,

$$\mathbf{A}' = \mathbf{A} - \nabla\theta, \qquad \nabla \times \mathbf{A}' = \mu_0 \mathbf{H} \neq 0 \tag{101}$$

as we expect the magnetic field $\mathbf{B} = \mu_0 \mathbf{H}$ present. Find the *diamagnetic* current,

$$\mathbf{j} = \frac{\delta L}{\delta \mathbf{A}'} = -\frac{n_f e^2}{m} \mathbf{A}'\,. \tag{102}$$

From Maxwell equations,

$$\nabla \times \mathbf{H} = \mathbf{j}\,, \Rightarrow \nabla \times (\nabla \times \mathbf{H}) = -\nabla^2 \mathbf{H} = \nabla \times \mathbf{j} \tag{103}$$

Insert (102) into (103),

$$\nabla^2 \mathbf{H} = \frac{n_f e^2 \mu_0}{m} \mathbf{H} \tag{104}$$

which is the well-know London equation, with London penetration depth

$$\begin{aligned} \lambda_L &= \sqrt{\frac{m}{n_f e^2 \mu_0}} \qquad \text{in SI units - this note} && (105) \\ &= \sqrt{\frac{mc^2}{4\pi n_f e^2}} \qquad \text{in Gauss units} && (106) \end{aligned}$$

The expression of λ_L can be found in [Tinkham, 1996, 2nd Ed., Page 6, Eq.(1.9); or Fetter and Walecka, Page 422, eq. (49.13)].

Note that the usual expression of London penetration depth is in terms of the superfluid density n_s in place of the total fermion density n_f. In BCS theory, the two are related as follows:

$$\frac{n_s(T)}{n_f} = \begin{cases} 1 - \left(\frac{2\pi\Delta_0}{k_B T}\right)^{1/2} e^{-\Delta_0/k_B T}\,, & T \to 0 \\ 2\left(1 - \frac{T}{T_c}\right) & T \to T_c \end{cases} \tag{107}$$

[Fetter and Walecka, Page 460, eq. (52.34)].

Important limit. In the BCS model, the superfluid density at zero temperature limit is equal to the total electron desnity, not that of the condensed pairs.

Exercise 5: Show that an electronic many-body system described by the effective Lagrangian (97) is superconducting, i.e., has zero resistivity, independent of the details of the microscopic model.